evant parameters is the same, but the flow arrangement is different in each case. For the cocurrent flow arrangement (P1) the temperature cross is between fluids 2 and 3. In the case of the countercurrent flow arrangement the temperature cross does not exist, while for both countercurrent–cocurrent (P3) and cocurrent–countercurrent (P4) flow arrangements temperature crosses (both direct and indirect) exist. In order to determine the existence of the temperature cross without analysing the temperature distributions within a heat exchanger, one can use equation (8). It is worth noting that the calculation should include double precision.

CONCLUDING REMARK

A compact solution was obtained for the temperature distribution and temperature cross of a three-fluid heat exchanger with two thermal communications among the thermally unbalanced fluid streams. The analysis was conducted for any of four possible fluid flow arrangements.

Acknowledgements—The author acknowledges many fruitful conversations with Dr B. S. Baĉlić and Dr R. K. Shah.

REFERENCES

1. T. B. Morley, Exchange of heat between three fluids, *Engineer* **155**, 134 (1933).
2. H. Hausen, *Warmeubertragung in Gegenstrom, Gleichstrom und Kreutzstrom*, Chapter 49, pp. 242–246. Springer, Berlin (1976).
3. W. Okolo-Kulak, Trojczynnikowe wymienniki ciepla (Three-fluid heat exchangers), *Zeszyty Naukowe Polit. Slaskiej—Mechanika* **1**, 7–78 (1954).
4. G. D. Rabinovich, Steady-state heat transfer between three fluids in parallel flow in a recuperative heat exchanger, *Inzh.-fiz. Zh.* **4**, 37–43 (1961).
5. T. Sorlie, Three-fluid heat exchanger design theory, counter and parallel-flow, TR No. 54. Department of Mechanical Engineering, Stanford University (1962).
6. V. V. G. Krishnamurty, Heat transfer in multi-fluid heat exchangers, *Indian J. Tech.* **4**, 167–169 (1966).
7. J. Schneller, Berechnung von Dreikomponenten-Warmeaustauschsystemen, *Chem.-Ing.-Tech.* **42**, 1245–1251 (1970).
8. B. S. Baĉlić, S. Grujin and M. Pavlović, A three-fluid heat exchanger design method, Report No.07-281/3 TFMP, University of Novi Sad (October 1987).
9. W. M. Kays and A. L. London, *Compact Heat Exchangers*, 2nd Edn, pp. 19–20. McGraw-Hill, New York (1984).

Pergamon

Int. J. Heat Mass Transfer. Vol. 37, No. 14, pp. 2187–2192, 1994

0017-9310/94 $7.00+0.00

Conjugate convection from a sphere in a porous medium

SHIGEO KIMURA

Tohoku National Industrial Research Institute, Nigatake 4-2-1, Miyagino, Sendai 983, Japan

and

IOAN POP

University of Cluj, Faculty of Mathematics, R-3400 Cluj, CP 253, Romania

(*Received* 16 *September* 1993 *and in final form* 14 *January* 1994)

1. INTRODUCTION

Transport processes through porous media is a subject that has been widely studied in the scientific community during the last two decades. This interest is justified by the important role it plays in the industrial sector, particularly in the insulating systems for buildings and heat exchanger devices, energy storage systems, material processing and geothermal systems. An excellent review on this subject was recently provided by Nield and Bejan [1].

Studies of convective heat transfer from an isothermal sphere embedded in a porous medium are important in many engineering and geophysical applications such as spherical storage tanks, packed beds of spherical bodies, solidification of a magma chamber and others. However, only a little work has been devoted to this problem in the past. An early paper by Yamamoto [2] presents an analytical solution for small Rayleigh numbers. This paper has recently been extended by Sano and Okihara [3] to the case of an unsteady convective flow. But boundary-layer solution (large Rayleigh numbers) of natural convection about a general axisymmetric heated body embedded in a porous medium have been presented by several authors, notably Merkin [4], Nilson [5] and Nakayama and Koyama [6]. In particular, Cheng [7] and Chen and Chen [8] have treated the case of a sphere. It was shown in [7] that this problem admits a similarity solution. Further, a systematic analysis of the problem of natural convection from an isothermal sphere immersed in a fluid-saturated porous medium has been presented by Pop and Ingham [9]. In addition to obtaining a second-order boundary-layer solution they used a finite-difference scheme to obtain numerical results for small values of the Rayleigh numbers, as well.

However, to the authors' knowledge the conjugation features of this problem have never been analysed. It is important to mention that conjugate heat transfer problems, in which the convective heat transfer depends strongly on the thermal boundary conditions, are important in many heat transfer equipments because this dependence usually degrades the heat exchanger performance. Hence, the present problem might have some relevance to understanding of a charging or discharging process of energy in regenerative

NOMENCLATURE

- a radius of the sphere
- a_c radius of the core region
- g acceleration due to gravity
- k thermal conductivity ration, k_s/k_f
- k_f effective thermal conductivity of the porous medium
- k_s thermal conductivity of the solid sphere
- K permeability of the porous medium
- M a total number of grid points in the radial direction
- N a total number of grid points in the angular direction
- $\overline{Nu}$ average Nusselt number based on ΔT
- $\overline{Nu}^*$ average Nusselt number based on $\overline{T_b}-T_\infty$
- q'' heat flux per unit area
- $\overline{q''}$ average heat flux per unit area
- r dimensionless radial coordinate, r'/a
- R radius ratio, a_c/a
- Ra Rayleigh number for porous medium based on ΔT, $g\beta\Delta TKa/\alpha\nu$
- Ra^* Rayleigh number for porous medium based on $\overline{T_b}-T_\infty$, $g\beta(\overline{T_b}-T_\infty)Ka/\alpha\nu$
- T temperature
- $\overline{T_b}$ average temperature at the surface of the sphere
- T_f temperature in the convective fluid
- T_s temperature in the solid sphere
- T_c, T_∞ temperatures (constant) of the core region and ambient fluid, respectively
- ΔT applied temperature difference, T_c-T_∞.

Greek symbols

- α thermal diffusivity
- β coefficient of thermal expansion
- δ boundary-layer thickness
- θ dimensionless temperature, $(T_b-T_\infty)/\Delta T$
- $\Delta\theta$ temperature increment
- ν kinematic viscosity
- σ dimensionless conjugate parameter, equation (15)
- ϕ angular coordinate
- ψ dimensionless streamfunction
- $\Delta\psi$ streamline increment.

Subscripts

- b boundary temperature
- f variables in the fluid-porous medium
- s variables in the solid sphere
- ij nodal points.

Superscripts

- $'$ dimensional variables
- - average quantities
- n iteration order.

porous bodies. On the other hand, it is worth pointing out that spherical shapes of canisters have been proposed for nuclear waste disposal in subseabeds.

The present paper is therefore concerned with the problem of conjugate natural convection about a sphere of thermal conductivity k_s, which is imbedded in a fluid-saturated porous medium of thermal conductivity k_f and of constant temperature T_∞. It is assumed that the sphere has a heated core region of a uniform temperature T_c, where $T_c > T_\infty$. Heat moves through the sphere by two-dimensional conduction and is transferred from the solid–porous matrix interface by natural convection to the ambient fluid-porous medium. Based on the full two-dimensional analysis, we are able to obtain accurate finite-difference solutions over the wide ranges of the parameters entering the problem. In addition, very simple but accurate asymptotic formulae were obtained for the average boundary temperature, and local and average Nusselt numbers when the Rayleigh number is very small ($Ra \to 0$) and very large ($Ra \gg 1$) in terms of the proper dimensionless variables. This is because, for the convenience of engineering applications, simple analytical and correlation equations are first preferable. A single dimensionless parameter, similar to the Biot number, is proposed, with which all the numerical results are nicely correlated.

2. BASIC EQUATIONS

Consider the problem of a steady natural convection from a sphere of radius a embedded in a fluid-saturated porous medium of uniform temperature T_∞. The sphere has a heated core region of radius a_c with $a_c < a$, and a uniform temperature T_c, where $T_c > T_\infty$. Both the fluid motion and temperature fields are axially symmetric and hence independent of the azimuthal coordinate. As a result, the physical system to be analysed here may be represented by the simple geometry shown in Fig. 1, where ϕ is measured clockwise from the vertically up position. Consideration will be then confined to the range $0 \leqslant \phi \leqslant \pi$ only. In terms of dimensionless variables, the problem is described by the following equations:

for the fluid-porous medium part [9]:

$$D^2\psi = r\sin\phi\left(\frac{\partial\theta_f}{\partial r}\sin\phi + \frac{\partial\theta_f}{\partial\phi}\frac{\cos\phi}{r}\right), \tag{1}$$

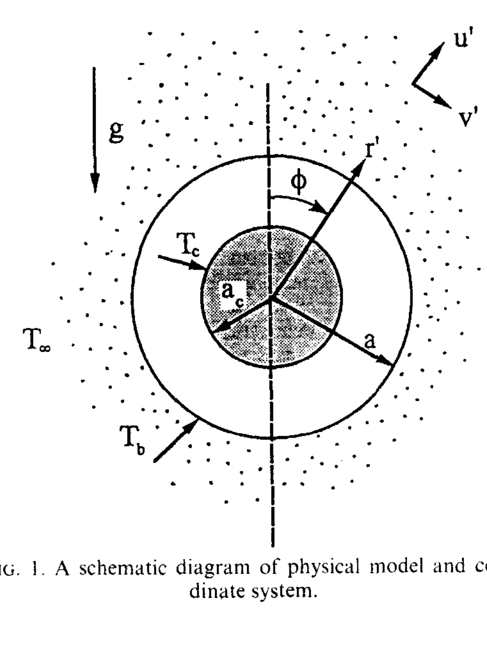

FIG. 1. A schematic diagram of physical model and coordinate system.

$$D^2\theta_f+2\frac{\cos\phi}{r^2}\frac{\partial\theta_f}{\partial\phi}+\frac{2}{r}\frac{\partial\theta_f}{\partial r}=\frac{Ra}{\sin\phi}\left(\frac{\partial\psi}{\partial\phi}\frac{\partial\theta_f}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial\theta_f}{\partial\phi}\right);\quad (2)$$

for the solid part:

$$D^2\theta_s+2\frac{\cos\phi}{r^2}\frac{\partial\theta_f}{\partial\phi}+\frac{2}{r}\frac{\partial\theta_f}{\partial r}=0,\quad (3)$$

where:

$$D^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}-\frac{\cot\phi}{r^2}\frac{\partial}{\partial\phi}.\quad (4)$$

Equations (1)–(3) are to be solved with respect to the following boundary conditions: at the fluid–solid interface:

$$\psi=0\quad \theta_f=\theta_s\ \frac{\partial\theta_f}{\partial r}=k\frac{\partial\theta_s}{\partial r}\quad \text{at } r=1;\quad (5a)$$

within the core region:

$$\theta_s=1\quad \text{at } r=R;\quad (5b)$$

far away from the sphere:

$$\psi=0\quad \theta_f=0\quad \text{as } r\to\infty;\quad (5c)$$

symmetric conditions:

$$\psi=\frac{\partial\theta_f}{\partial\phi}=\frac{\partial\theta_s}{\partial\phi}=0\quad \text{on } \phi=0,\pi.\quad (5d)$$

The average Nusselt number can be expressed as [9]:

$$\overline{Nu}=\frac{1}{2}\int_0^\pi\left(-\frac{\partial\theta_f}{\partial r}\right)_{r=1}\sin\phi\,\mathrm{d}\phi.\quad (6)$$

3. ANALYTICAL SOLUTIONS

The next step is to solve equations (1)–(3) subject to the boundary conditions (5). As already mentioned, we shall first present approximate analytical solutions of these equations, which are based on our recent work on conjugate natural convection problems [11, 12].

3.1. *Flow at low Rayleigh number*

When the Rayleigh number is small ($Ra\to 0$), the solutions can be assumed a series expansion in powers of Ra:

$$(\psi,\theta_f,\theta_s)=(\psi^0,\theta_f^0,\theta_s^0)+(\psi^1,\theta_f^1,\theta_s^1)Ra+\cdots.\quad (7)$$

Working from the leading order terms (conduction solution) and imposing the boundary conditions (5a)–(5d), we obtain the following solutions of flow and temperature fields:

$$\psi^0=\frac{1}{2}\frac{kR}{(k-1)R+1}\left(r-\frac{1}{r}\right)\sin^2\phi,\quad (8)$$

$$\theta_f^0=\frac{kR}{(k-1)R+1}\frac{1}{r},\quad (9a)$$

$$\theta_s^0=\frac{R}{(k-1)R+1}\left(k-1+\frac{1}{r}\right).\quad (9b)$$

For practical considerations, it is convenient to introduce the dimensionless average boundary temperature $\theta_b=(T_b-T_\infty)/\Delta T$, where T_b is the average temperature distribution at the surface of the sphere and is defined as:

$$\overline{T_b}=\frac{1}{2}\int_0^\pi T_f(r',\phi)_{r'=a}\sin\phi\,\mathrm{d}\phi.\quad (10)$$

Thus from (9a), in the limit of $Ra\to 0$, we get:

$$\overline{\theta_b}=\frac{kR}{(k-1)R+1}.\quad (11)$$

3.2. *Flow at large Rayleigh number*

In order to obtain analytical boundary-layer solutions for this problem, we use the fact that at the solid–fluid interface ($r'=a$) the heat fluxes from the solid part and through the boundary-layer are equal. Within a strictly one-dimensional analysis, this leads to:

$$q''=\frac{k_s(T_c-\overline{T_b})}{a^2\left(\frac{1}{a_c}-\frac{1}{a}\right)}=k_f\frac{\overline{T_b}-T_\infty}{\delta},\quad (12)$$

where $\delta\sim a/\overline{Nu}=1.9516aRa^{-1/2}$ is the thickness of the boundary-layer near the sphere. Here the average Nusselt number $\overline{Nu}=0.5124Ra^{1/2}$ is taken from Pop and Ingham [9]. Thus, from (12), we obtain the following approximate solution for the dimensionless average boundary temperature:

$$\overline{\theta_b}=\frac{1.9561kR}{1.9561kR+(1-R)Ra^{1/2}},\quad (13)$$

or:

$$\overline{\theta_b}=\frac{1}{1+\sigma},\quad (14)$$

where σ is the Biot number like parameter, which is given for the present problem by:

$$\sigma=\left(\frac{1}{R}-1\right)Ra^{1/2}/1.9516k.\quad (15)$$

Since a similar non-dimensional parameter arises for all the convection–conduction conjugate heat transfer problems, it would be convenient to call this dimensionless constant *conjugate modulus*. This is essentially the same parameter called wall thickness parameter by Bejan and Anderson [17].

It should be, however, noted that, in a strict sense, the Rayleigh number in equation (13) is not the one which actually drives the flow. Following the argument suggested by Vynnycky and Kimura [10], we define the Rayleigh number Ra^* based on $(\bar{T}_b-T_\infty)$; the real temperature difference that drives the flow, i.e. $Ra^*=g\beta K(\overline{T_b}-T_\infty)/\alpha\nu=\theta_b Ra$. Thus, using again equation (12), we have:

$$\frac{k_s(T_c-\overline{T_b})}{a^2\left(\frac{1}{a_c}-\frac{1}{a}\right)}=k_f\frac{\overline{T_b}-T_\infty}{1.9516k}\left(\frac{\overline{T_b}-T_\infty}{\Delta T}\right)^{1/2}Ra^{*1/2},\quad (16)$$

which, after some algebra, reduces to:

$$\sigma\overline{\theta_b}^{3/2}+\overline{\theta_b}-1=0.\quad (17)$$

This is a cubic equation of $\overline{\theta_b}^{1/2}$, and it has only one real root in a range between 0 and 1. The physically relevant root of this equation is given in ref. [10].

Further, we note that the average heat flux through the boundary-layer can be expressed as:

$$\overline{q''}=\frac{k_f}{a}(\overline{T_b}-T_\infty)\overline{Nu^*}=0.5124\frac{k_f}{a}(\overline{T_b}-T_\infty)Ra^{*1/2},\quad (18)$$

where $\overline{Nu^*}$ is the average Nusselt number based on $(\bar{T}_b-T_\infty)$, which is given by $\overline{Nu^*}=0.5124Ra^{*1/2}$. Therefore, the over-all average Nusselt number through the solid and the boundary-layer:

$$\overline{Nu}=\frac{\overline{q''}a}{k_f\Delta T},\quad (19)$$

can be derived from the average heat flux expression as given by equation (18) and the average boundary temperature equations (13) or (17). The end result is:

$$\overline{Nu}=0.5124\overline{\theta_b}^{3/2}Ra^{1/2}.\quad (20)$$

Finally, if we eliminate Ra between (15) and (20), one gets:

$$\overline{Nu}\left(\frac{1}{R}-1\right)\Big/k\bar{\theta}_b^{3/2} = \sigma, \quad (21)$$

indicating that the average Nusselt number is also correlated by a single parameter σ.

4. NUMERICAL COMPUTATION

The method chosen for the numerical solution of the governing partial differential equations (1)–(3) subject to the boundary conditions (5) was solved by the finite-differences and coordinate transformation grid network as described by Küblbeck *et al.* [13]. The grid net work due to coordinate transformation has some advantage over intuitive grid generation. This method produces a smooth stretch of grid spaces from fine grids near the sphere surface to coarse ones in the far field. This is particularly convenient when the far field boundary conditions are involved. The advection terms in the energy equation are discretized by a second-order upwind-type approximation. The total number of the nodal points varied from 31 × 60 (31 in the angular direction and 60 in the radial direction) to 41 × 90 depending upon the parameters R and Ra. In Table 1 we show the values of

Table 1. Comparison of numerical results with the boundary-layer calculations ($k \to \infty$)

Ra	$\overline{Nu}$ Boundary-layer theory (Pop and Ingham [9])	Present results with 31 × 54
40	3.2407	3.6093
100	5.1240	5.3752
200	7.2464	7.2320

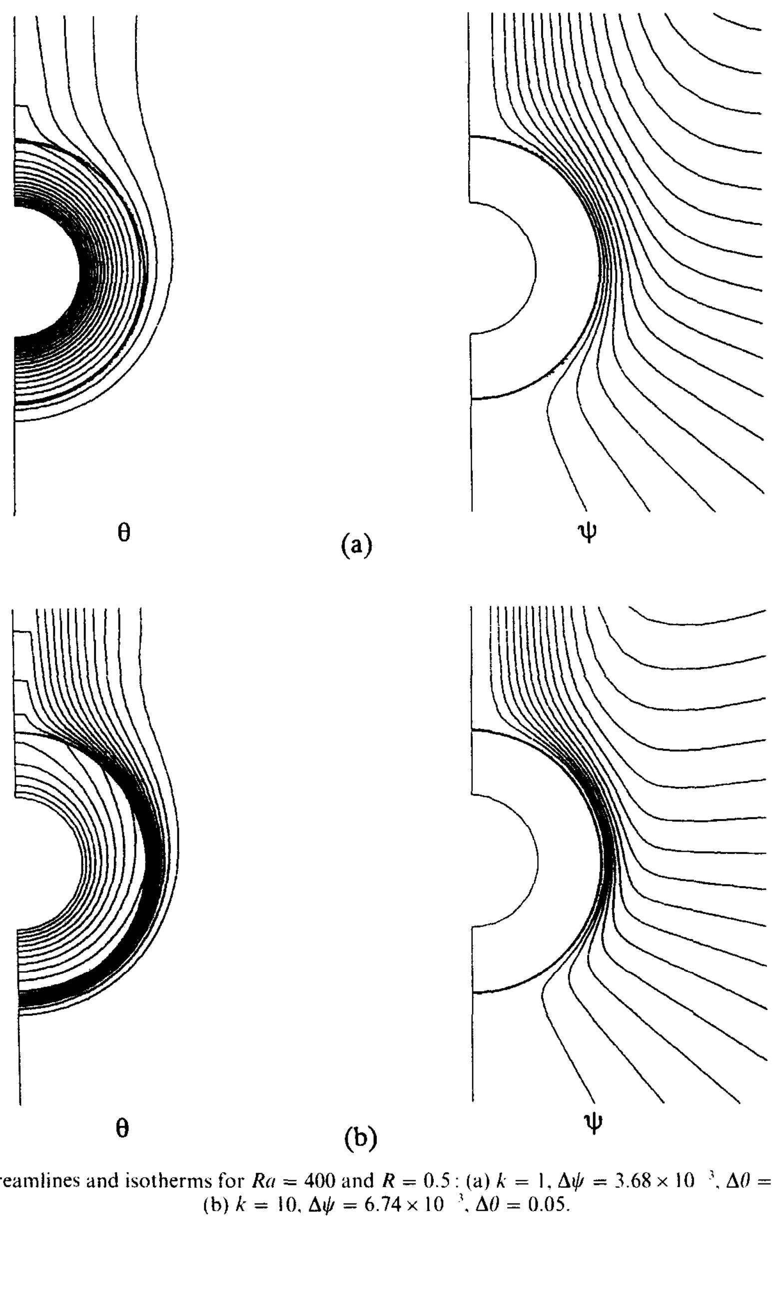

FIG. 2. Streamlines and isotherms for $Ra = 400$ and $R = 0.5$: (a) $k = 1$, $\Delta\psi = 3.68 \times 10^{-3}$, $\Delta\theta = 0.05$; and (b) $k = 10$, $\Delta\psi = 6.74 \times 10^{-3}$, $\Delta\theta = 0.05$.

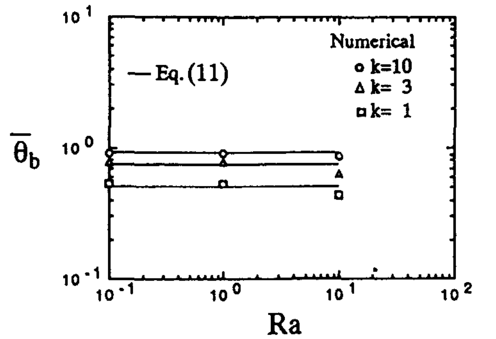

FIG. 3. Variation of the dimensionless average boundary temperature with small values of Ra for $R = 0.5$.

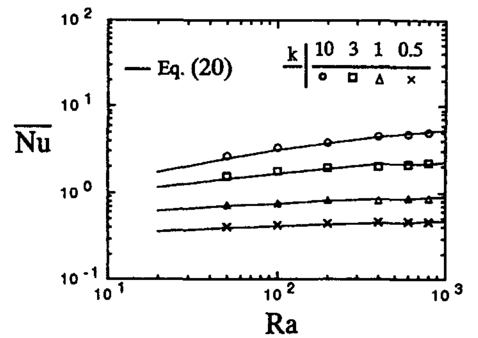

FIG. 5. Variation of the average Nusselt number with large values of Ra for $R = 0.5$.

average Nusselt number for $k \to \infty$ and the boundary-layer approximation. It is seen that as Ra increases the numerical Nusselt numbers approach the values predicted by the boundary-layer approximation. The size of the computational domain is taken as roughly 5–10 times the sphere radius. With the open boundary conditions this proves large enough to produce the size-independent results, except for very low Rayleigh numbers, where the thermal diffusion from the sphere penetrates far deeper than the cases of high Rayleigh numbers. The convergence of the numerical results is established locally based on the criterion, for instance:

$$\frac{\sum_i^M \sum_j^N |\theta_{i,j}^{n+1} - \theta_{i,j}^n|}{\sum_i^M \sum_j^N |\theta_{i,j}^{n+1}|} < 10^{-6}, \tag{22}$$

where the superscript n denotes the iteration order. For other physical quantities the same criterion is employed.

The dimensionless parameters in the present study are the Rayleigh number Ra, the thermal conductivity ratio k, radius ratio R and the conjugate modulus σ. The values of these parameters are $Ra = 0.1$, 1, 10, 40, 100, 200, 400 and 1000; $k = 0.5$, 1, 3 and 10; $R = 0.5$; $\sigma = 0.1$, 1, 10 and 100.

5. RESULTS AND DISCUSSION

Typical computed results for streamlines and isotherms are displayed in Fig. 2 for $Ra = 400$; $k = 1$ and 10; $R = 0.5$. Each curve in the plots on the left-hand side represents an isotherm line while each curve on the right-hand side represents a streamline. It is seen that the boundary-layer becomes thinner with increasing k. As is expected, evidence of plume development is found near the top surface of the sphere.

Next, variation of the dimensionless average boundary temperature and average Nusselt number with Ra is shown in Figs. 3–5. The analytical solutions (11), (13), and (20) have also been included for reference. It is noticed that very good agreement exists between these analytical solutions and fully two-dimensional numerical results in this problem. Further we see that $\overline{\theta}_b$ and $\overline{Nu}$ are substantially influenced by the conjugate parameter k; they increase as k increased. However, as Fig. 3 shows, $\overline{\theta}_b$ remains constant for small values of Ra, i.e. curves are linear and flat. In particular, they become close to one for large values of $k(\sim 10)$. It suggests that the boundary and the core temperatures are nearly equal. On the other hand, for comparatively smaller values of k these temperatures become unequal.

Results from equations (14), (17) and (21) are finally presented together with the numerical ones. In Fig. 6, showing the comparison of $\overline{\theta}_b$ for both analytical equations (14) and (17), we conclude that the agreement between these two-model equations is good, particularly for small values of the parameter σ. A very good agreement between analytical predictions and numerical ones is also seen over a wide rage of σ in Fig. 7. It suggests that the present conjugate problem depends only on the single parameter σ, i.e. the Biot number like parameter and may be termed as *conjugate modulus* in the present paper. It should be recalled that the *conjugate modulus* may take slightly different forms depending upon the geometries involved (for example, see Vynnycky and Kimura [14], Bejan and Anderson [15]). Nonetheless, no doubt, this greatly simplifies the study of this otherwise complicated heat transfer model.

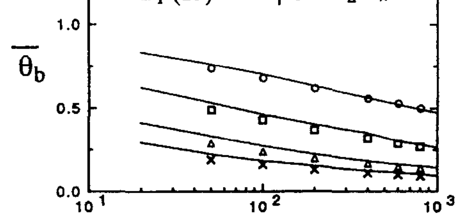

FIG. 4. Variation of the dimensionless average boundary temperature with large values of Ra for $R = 0.5$.

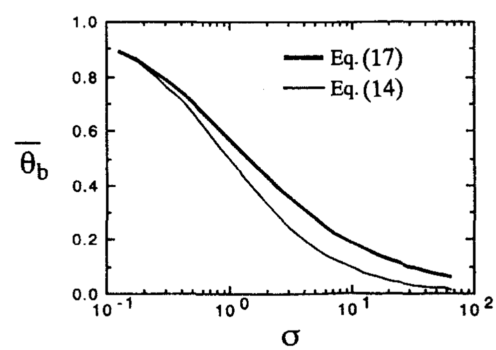

FIG. 6. Variation of the dimensionless average boundary temperature with σ.

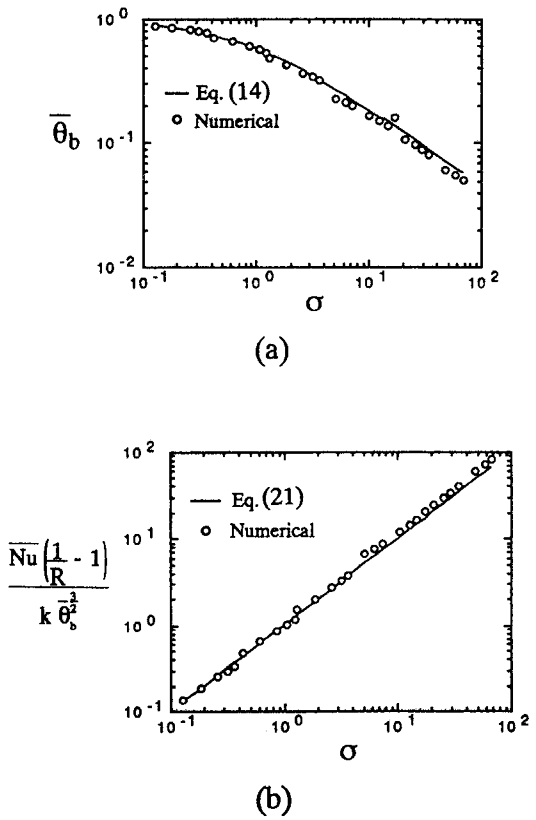

FIG. 7. Variation of: (a) the dimensionless average boundary temperature; and (b) the expression $\overline{Nu}(1/R-1)/k\overline{\theta}_b^{3/2}$ with σ.

Acknowledgement—The second author (IP) wishes to express his gratitude to the Tohoku National Industrial Research Institute for their kind hospitality during the autumn of 1992 when the authors started this study.

REFERENCES

1. D. A. Nield and A. Bejan, *Convection in Porous Media*. Springer, New York (1992).
2. K. Yamamoto, Natural convection about a heated sphere in a porous medium, *J. Phys. Soc. Japan* **37,** 1164–1166 (1974).
3. T. Sano and R. Okihara, Natural convection about a sphere immersed in a porous medium at small Rayleigh numbers, *Fluid Dynamics Res.* **13,** 39–44 (1994).
4. J. H. Merkin, Free convection boundary layers on axi-symmetric and two-dimensional bodies of arbitrary shape in a saturated porous medium, *Int. J. Heat Mass Transfer* **22,** 1461–1462 (1979).
5. R. H. Nilson, Natural convection boundary layer on two-dimensional and axisymmetric surfaces in high-Pr fluids or in fluid saturated porous media, *J. Heat Transfer* **103,** 803–807 (1981).
6. A. Nakayama and H. Koyama, Free convection heat transfer over a nonisothermal body of arbitrary shape embedded in a fluid-saturated porous medium, *J. Heat Transfer* **109,** 125–130 (1987).
7. P. Cheng, Natural convection in a porous medium: external flows. In *Natural Convection: Fundamentals and Applications* (Edited by S. Kakac *et al.*), pp. 475–513. Hemisphere, Washington, DC (1985).
8. H. T. Chen and C. K. Chen, Natural convection of a non-Newtonian fluid about a horizontal cylinder and a sphere in a porous medium, *Int. Comm. Heat Mass Transfer* **15,** 605–614 (1988)
9. I. Pop and D. B. Ingham, Natural convection about a heated sphere in a porous medium, *Proc. 9th Int. Heat Transfer Conf.* **2,** pp. 567–572. Hemisphere, New York (1990).
10. M. Vynnycky and S. Kimura, Conjugate free convection due to a vertical plate in a porous medium, *Int. J. Heat Mass Transfer* **37,** 229–236 (1944).
11. S. Kimura and I. Pop, Conjugate free convection from a circular cylinder in a porous medium, *Int. J. Heat Mass Transfer* **35,** 3105–3113 (1992).
12. S. Kimura and I. Pop, Conjugate natural convection from a horizontal circular cylinder, *Numer. Heat Transfer A* **25,** 347–361 (1994).
13. K. Küblbek, G. P. Merker and J. Straub, Advanced numerical computation of two-dimensional time-dependent free convection in cavities, *Int. J. Heat Mass Transfer* **23,** 203–217 (1980).
14. M. Vynnycky and S. Kimura, Conjugate free convection due to a heated vertical plate, submitted to *Numer. Heat Transfer*.
15. A. Bejan and R. Anderson, Heat transfer across a vertical impermeable partition imbedded in a porous medium, *Int. J. Heat Mass Transfer* **24,** 206–209 (1981).